4.5.3 Decompositions with 2 summands

① Describe some decomposable braided V.S. of rack type with f.d. Nichols algebra

② state the main result of [53]

Example 37. Let $X = \mathcal{D}_4 = I_2 \,{}_{\sigma}\!\amalg_{\sigma} I_2$, $\sigma \neq id$

(Recall P16/Exercise 11, P15/Example 10,

A set $X \neq \emptyset$, $\forall x, y \in X$, $x \triangleright y = y$, X is a Rack, X is called abelian.

Pairs (ξ, w) of morphisms of rack $\xi : Y \to Aut\, Z$, $w : Z \to Aut\, Y$

s.t. ① $y \triangleright w_z(u) = w_{\xi_y(z)}(y \triangleright u)$, $\forall y, u \in Y$, $z \in Z$,

$\left(\xRightarrow{\text{abelian}} w_z = w_{\xi_y}(z) \right)$

② $z \triangleright \xi_y(w) = \xi_{w_z(y)}(z \triangleright w)$, $\forall y \in Y$, $z, w \in Z$,

$\left(\xRightarrow{\text{abelian}} \xi_z = \xi_{w_z}(y) \right)$

Y, Z are two racks, $X = Y \dot{\cup} Z$

The rack X is denoted $Y_\xi \amalg_w Z$

$y \triangleright z = \xi_y(z)$, $y \in Y$, $z \in Z$,

$z \triangleright y = w_z(y)$, $y \in Y$, $z \in Z$)

concretely, $X = \{1, 2\}_{(34)} \amalg_{(12)} \{3, 4\}$

Then $kX = V_1 \oplus V_2$, where V_1 is spanned by $(x_i)_{i \in I_2}$, while V_2 is spanned by $(x_j)_{j \in I_{3,4}}$

Let $p, q, r, t \in k^\times$, $p \neq 1 \neq q$, and $\varepsilon, \varepsilon' \in G_2$, Define a braiding on kX by

$c|_{V_1\otimes V_1}$ is of diagonal type with matrix $\begin{pmatrix} q & \varepsilon q \\ \varepsilon q & q \end{pmatrix}$

$c|_{V_2\otimes V_2}$ is of diagonal type with matrix $\begin{pmatrix} p & \varepsilon' p \\ \varepsilon' p & p \end{pmatrix}$,

$$\left(c(x_i\otimes x_j)_{i\in I_2,\, j\in I_{3,4}}\right) = \begin{pmatrix} x_4\otimes x_1 & t^2 x_3\otimes x_1 \\ \varepsilon' x_4\otimes x_2 & \varepsilon' t^2 x_3\otimes x_2 \end{pmatrix}, \qquad (87)$$

$$\left(c(x_j\otimes x_i)_{j\in I_{3,4},\, i\in I_2}\right) = \begin{pmatrix} x_2\otimes x_3, & r^2 x_1\otimes x_3 \\ \varepsilon x_2\otimes x_4, & \varepsilon r^2 x_1\otimes x_4 \end{pmatrix}$$

~~Now $\overset{-1}{\circ}\overset{-1}{\text{———}}\overset{-1}{\circ}$ is a Dynkin diagram of Cartan~~

Exercise 31.1. Prove that this is indeed a braiding

Prove: (For example)

$$(c\otimes id)(id\otimes c)(c\otimes id)(x_1\otimes x_4\otimes x_3)$$
$$= (c\otimes id)(id\otimes c)(t^2 x_3\otimes x_1\otimes x_3)$$
$$= (c\otimes id)(t^2 x_3\otimes x_4\otimes x_1)$$
$$= \varepsilon' p t^2 x_4\otimes x_3\otimes x_1$$

$$(id\otimes c)(c\otimes id)(id\otimes c)(x_1\otimes x_4\otimes x_3)$$
$$= (id\otimes c)(c\otimes id)(\varepsilon' p\, x_1\otimes x_3\otimes x_4)$$
$$= (id\otimes c)\ (\varepsilon' p\, x_4\otimes x_1\otimes x_4)$$
$$= \varepsilon' p t^2 x_4\otimes x_3\otimes x_1$$

$$\therefore (c\otimes id)(id\otimes c)(c\otimes id)(x_1\otimes x_4\otimes x_3)$$
$$= (id\otimes c)(c\otimes id)(id\otimes c)(x_1\otimes x_4\otimes x_3)$$

Other cases ($x_1\otimes x_1\otimes x_1$, $x_1\otimes x_1\otimes x_2$, ..., $x_4\otimes x_4\otimes x_4$) are similar

Rmk: by P13 / Equation (25), $c(x,y) = (x\triangleright y, x)$

so, to give $\triangleright$ ~~and~~ is equivalent to give c (87)

2. Assume that $\varepsilon = \varepsilon' = 1$, consider the basis $(y_h)_{h \in I_4}$ of kX where $y_1 = rx_1 + x_2$, $y_2 = -rx_1 + x_2$, $y_3 = tx_3 + x_4$,

$y_4 = -tx_3 + x_4$

Then c on this basis is of diagonal type with matrix

$$\begin{pmatrix} q & q & t & -t \\ q & q & t & -t \\ r & -r & p & p \\ r & -r & p & p \end{pmatrix}$$

Verify: $c(y_2 \otimes y_3) = c(-rx_1 \otimes tx_3 - rx_1 \otimes x_4 + x_2 \otimes tx_3 + x_2 \otimes x_4)$

$= -rt\, x_4 \otimes x_1 - rt^2\, x_3 \otimes x_1 + t\varepsilon'\, x_4 \otimes x_2 + \varepsilon' t^2\, x_3 \otimes x_2$

$= \cancel{t(rx_4)} - tr(x_4 + tx_3) \otimes x_1 + \varepsilon' t(x_4 + tx_3) \otimes x_2$

$= t(x_4 + tx_3) \otimes (\varepsilon' x_2 - rx_1)$ (by $\varepsilon' = 1$)

$= t\, y_3 \otimes y_2$

Others, such as $y_1 \otimes y_1$, $y_1 \otimes y_2$, $\cdots$ $y_4 \otimes y_4$ are similar

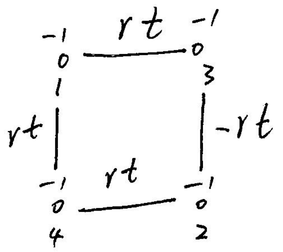
Prop: If $\dim \mathfrak{B}(V) < \infty$, then $p = q = -1$

Pf: by P39/Example 31, $p, q \in G_2' \cup G_3'$

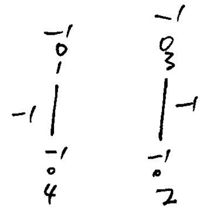
by [D45], $p, q \notin G_3'$

$\therefore p = q = -1$

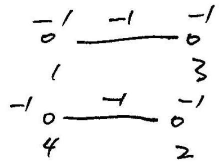
The Dynkin diagram is



if $rt = 1$:



if $rt = -1$:



$\because$  is a Dynkin Diagram of Cartan type A_2 at -1

By elementary arguments, (Bialgebras of type one, 1978, Nichols) its Nichols algebra has dim 8.

$\therefore$ if $rt \in G_2$, then $\dim \mathfrak{B}(V) = 64$.

If $rt \notin G_2$, then $\dim \mathfrak{B}(V) = \infty$ by [45]

3. If $\varepsilon = \varepsilon' = -1$, then there is a twist ϕ as in Example 30 that reduces to the previous one

(P37/example 30. Let q and q' be 2-cocycles on X.
We say that q and q' are twist-equivalent
if there exists $\phi : X \times X \to k^\times$.
s.t. $q' = q^\phi$
$q^\phi_{xy} = \phi(x, y)\, \phi^{-1}(x \triangleright y, x)\, q_{xy}$, $x, y \in X$.
if q and q' are twist-equivalent,
the Hilbert series of $\mathfrak{B}(X, q)$ and $\mathfrak{B}(X, q')$ coincide

1. Preliminaries

Group, tensor algebra → symmetric algebra ($S(V)$, $\wedge(V)$)

↓

enveloping algebra

coalgebra and Hopf algebra

tensor coalgebra

GK-dim

2. Braided V.S. :
- symmetries
- Diagonal type (include Hecke type)
- triangular type
- Rack type (Racks)

tensor categories

↓

Braided tensor categories

↓

YD modules

3. Hopf algebras in Braided tensor categories

Bosonization

Nichols algebras

techniques
- Direct computation
- Dual
- twisting
- discard
- Decomposition

4. Classes of Nichols algebras

① symmetries and Hecke type : solved

② Diagonal type : solved

③ triangular type (finite GK-dim over abelian group)

④ Rack type : infinite dim

- Criteria of types C, D, F
- Alternating and symmetric groups
- Finite simple groups of Lie type
- Sporadic groups

⑤ Rack type : finite dim

FK algebra

f.d. Nichols algebras of some affine racks

Decompositions with 2 summands

5. Unsolved problem

① other classes of Nichols algebra

② finite GK-dim over non-abelian group

③ conjugacy class of a finite group, others ?

④ f.d. Nichols algebra of other affine racks